Analysis of experimental data

Irina Novikova

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1 Absolute and relative uncertainties

Main goal of any experiment is to measure some physical quantity. The measurements can be direct or indirect. In a *direct* measurement one measures directly the value of interest. The examples of direct measurements are the measurements of object length with a ruler, or the measurements of electric current with an amperemeter. Much more often measurements are *indirect*; in this case the main value is calculated based on the direct measurements of supplementary values: for example, an unknown resistance may be calculated through the measurement of a current and a voltage drop across it, or the volume of the cube is calculated by multiplying the measured lengths of its sides.

The result of any measurement (both direct and indirect) gives only an approximate value of the actual physical quantity. How accurate a measurement is depends on several parameters, such as, for example, accuracy of measuring devices. Indeed, a caliper provides much more accurate measurements than a regular ruler. Since a result of any measurement is only approximate, any calculation based on this approximate measured value is also approximate. That is why for any direct or indirect measurement it is necessary to include the value of an **absolute uncertainty**. For example:

$$L = (24.7 \pm .4)\text{cm} \tag{1}$$

For most cases an absolute uncertainty is rounded up to only one significant figure. The only exception is when this figure is "1" - in this case the second significant figure is allowed. Also, the number of significant figures in a measured value notation is limited by its absolute uncertainty. The value must be rounded up such that the last significant figure is in the same decimal register as the last significant figure of the absolute uncertainty. Some examples:

Incorrect	Correct
3.45 ± 0.5 m	3.5 ± 0.5 m
75.742 ± 0.12 kg	75.74 ± 0.12 kg
8.8 ± 0.01 mm	8.80 ± 0.01 mm
2504 ± 40 nm	$(250 \pm 4) \times 10^1$ nm or $(2.5 \pm 0.4) \times 10^2$ nm

It is a good practice to use a proper number of significant figures for each experimental value in records. Sometimes a physical value may be written without absolute uncertainty - in this case the absolute uncertainty is ± 1 in the last significant figure. For example, the is a difference between the notation 7000 m and 7×10^4 m: in the first case the measured length has 4 significant figures, so it is defined with accuracy 1 m - (7000 ± 1) m; in the second case there is only one significant figure, so the length is $(7 \pm 1) \times 10^4$ m.

Notice that absolute uncertainty has the same unit as a measured value. One can also define a *relative uncertainty*- a ratio between the absolute uncertainty and the measured value. A relative uncertainty is dimensionless, and it shows a relative accuracy of the measured result (or per cent error).

Bottom line: any physical value is measured with uncertainty; this uncertainty **must** be defined for any physical value measured or calculated in any experiment.

2 How to find an absolute uncertainty of a measured physical value

Let's first discuss an uncertainty of a direct measurement (*i.e.* when the physical value is measured directly). In this case there are two general classes of experimental errors: systematic and random. The boundary between these two is not well-defined. There are certainly some experimental uncertainties which are not either fully random or fully systematic.

Systematic uncertainties/errors: these errors appear and stay constant for any repetition of the experiment. A classic example of a systematic uncertainty is an uncalibrated measuring instrument: it does not matter how many times the experiment is repeated - the results will be consistently incorrect. The only way to correct a systematic error is to identify it and compensate for it in the measured results. For example, if the readings of a voltmeter are known to be 20% below the actual value, it is possible to account for that by increasing the measured values by 20%. However, accounting for a systematic error means guessing of its effect, so generally it is much better to redesign an experiments to avoid any systematic errors.

Random uncertainties/errors: these errors produce scatter in measured value, i.e. the results of a series of the experiments are somewhat different from one another. This variation may be due to some variations in experimental conditions, fluctuations in the measuring device performance, or human error. In electronic instruments it may be due to noise in electronic circuits. Random uncertainties may be reduced by repeating the experiments many times.

There are three main categories of random errors: ***Obvious errors*** or outliers are the results which are noticeably inconsistent with the rest of the data set. Mathematically we can suspect that a particular point is an outlier if omitting it reduces the standard deviation of the whole set by a lot - a factor of 3 or more. In this case it is wise not to include this point in the data analysis, providing that the number of remaining points is still sufficient.

However, it is very bad practice to just through away data points. If you suspect that there is an error in one measurements, it is the best to go back, re-examine the situation and find the source of error.

Instrumental errors come from a limited resolution of a measuring apparatus. Any reading is limited to a certain number of significant digits. This limit is usually obvious for analog devices with scales and divisions. Although there is no one set rule on how to define the instrumental uncertainty, the rule of thumb is that it is $\pm$ half of the smallers division. For example, a mass of a brick measured using a bathroom scale (the smallest division is 1 lb) is (12.5 ± 0.5) lb.

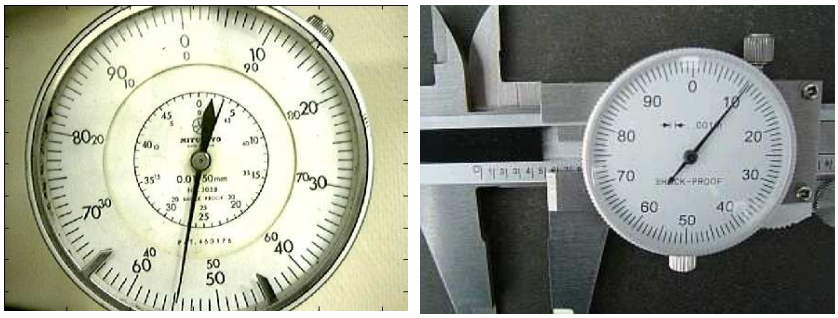

Figure 1: Examples of reading analog scales with instrumental uncertainties.

For digital read-outs it is safe to quote ± 1 in the least significant digit. For example, the correct way to write down a time measured using a digital stop watch (showing minutes and seconds) will be (36 ± 1) s.

If only one measurement is made, the precision of the measured value is limited by instrumental uncertainty, which must be quoted. Amazingly, if the same value measured many times, and the source of fluctuations is completely random, then it is possible to measure the average for the set with greater precision than the instrument allows.

Statistical errors become important when an experiment is repeated many time in nominally identical conditions, but the outcomes vary within some range around some average value. In this case the uncertainty of the average is calculated based on the statistical properties of the measured data set.

Let's suppose that some experiment was repeated N times, and N different outcomes for a value of interest x were recorded to the lab book as $\{x_i\}$. Then it is logical to assume that a true value of x should be close to the calculated average of the results of repeated measurements $\{x_i\}$:

$$x_{ave} = \frac{\sum_{i=1}^{N} x_i}{N} \tag{2}$$

We also need to determine the uncertainty of x_{ave}; we expect that the true value of the measured parameter lies within the range $[x_{ave} - \Delta x; x_{ave} + \Delta x]$.

The most accurate way to find Δx depends of the size N of the data set. We will have to distinguish two important cases: *small data sets* (number of data points is less than 5-10) and *large data sets*(number of data points is greater than 10-20).

Small data set

Essential quality of a small data set is that we cannot extract any true statistical information about the set because there are too few points. In this case we can only assume that each next measurement will fall in the range of the existing data spread; thus, the uncertainty of each individual measurement is

$$\sigma = \frac{x_{\max} - x_{\min}}{2} \tag{3}$$

However, the more measurements are made, the more precise the average value becomes, even though the uncertainty for each individual measurement stays constant. A properly defined uncertainty of the average value is called a *standard error of the mean* (SEM):

$$\Delta x_{ave} = \frac{\sigma}{\sqrt{N}} = \frac{x_{\max} - x_{\min}}{2\sqrt{N}} \tag{4}$$

Large data set

As the number of measurements increases, the range of the data spread may reach some stable value. However, larger number of points allows to apply more rigorous statistical analysis. It is known that for any data set with finite random fluctuations the probability distribution of outcomes approaches a Gaussian bell-shaped dependence as the number of points increases. The top of the bell gives the average value of the data set as defined in Eq.(2). The standard deviation determines the width of the probability distribution:

$$\sigma_N = \sqrt{\frac{\sum_{i=1}^{N}(x_i - x_{ave})^2}{(N-1)}} \tag{5}$$

Physically the standard deviation σ_N gives the uncertainty for each measurement, i.e. we can assume that the next measurement would fall within one standard deviation from the calculated average value. Similarly, the uncertainly of the mean value SEM decreases as the number of data points increases:

$$\Delta x_{ave} = \frac{\sigma_N}{\sqrt{N}} = \sqrt{\frac{\sum_{i=1}^{N}(x_i - x_{ave})^2}{(N-1)N}} \tag{6}$$

Technically the above method can be applied for small sets as well if the additional corrections (Student's coefficients) are added to the expression for the uncertainty.

No matter what method was used to analyze a data set, the final result of the measurement should be reported as:

$$x_{ave} \pm \Delta x,$$

where x_{ave} is the average value of the set, and Δx is the standard error of the mean, rounded up to one significant figure.

3 How to find an absolute uncertainty of a calculated physical value

Let me first introduce the general treatment of the propagation of errors, and then do a few examples.

Single-variable function

How to estimate the uncertainty ΔY of a calculated value $Y = f(x)$ if x is measured with uncertainty Δx? If the value of x is known to be within the range $[x - \Delta x; x + \Delta x]$, and $f(x)$ is monotonous in that region, than the expected range of values of Y is:

$$\Delta Y = \frac{1}{2}|f(x + \Delta x) - f(x - \Delta x)| \approx \left|\frac{df}{dx}\right| \Delta x \tag{7}$$

Using this rule, it is easy to figure out the uncertainty for some common cases:

1. Multiplication to the constant A:
$$Y = Ax; , \;\; \Delta Y = A\Delta x;$$

2. Power dependence:
$$Y = x^n; , \;\; \Delta Y = \left|nx^{n-1}\right| \Delta x = Y \cdot n\frac{\Delta x}{x};$$

Multi-variable function

Very often in data analysis there is a need to calculate a value Y and uncertainty of the value ΔY that depends on several experimental parameters $x_1, x_2, \ldots$. In this case the uncertainties of all these parameters have to be taken into account. An important assumption we have to make is that any fluctuations in these parameters are independent. In this case we can generalize Eq.(7) to the case of several variables:

$$\Delta Y = \sqrt{(\frac{\partial f}{\partial x_1}\Delta x_1)^2 + (\frac{\partial f}{\partial x_2}\Delta x_2)^2 + \ldots} \tag{8}$$

Although this expression looks cumbersome, it is very handy for finding expressions for known physical formulae:

1. Addition and subtraction:
$$Y = x_1 \pm x_2; , \;\; \Delta Y = \sqrt{\Delta x_1^2 + \Delta x_2^2};$$

2. Power dependence:
$$Y = x_1 \times x_2; , \;\; \Delta Y = \sqrt{x_2^2\Delta x_1^2 + x_1^2\Delta x_2^2}$$
or, changing it in more traditional form:
$$\Delta Y = Y\sqrt{(\frac{\Delta x_1}{x_1})^2 + (\frac{\Delta x_2}{x_2})^2}$$
It is also easy to show that the same expression is true for division:
$$Y = \frac{x_1}{x_2}; , \;\; \Delta Y = Y\sqrt{(\frac{\Delta x_1}{x_1})^2 + (\frac{\Delta x_2}{x_2})^2}$$

The equations above are sufficient for the majority of the physics problems. Let me just throw a few examples to illustrate how the equations for uncertainties work:

- Temperature difference:
$$T_{dif} = T_{fin} - T_{in}; , \;\; \Delta T_{dif} = \sqrt{(\Delta T_{fin})^2 + (\Delta T_{in})^2} = \sqrt{2}\Delta T$$
if the uncertainties $\Delta T_{fin} = \Delta T_{in} = \Delta T$.

- Speed of light pulse:

$$v = d/t;, \;\; \Delta v = v\sqrt{(\frac{\Delta d}{d})^2 + (\frac{\Delta t}{t})^2}$$

- Volume of a cylinder:

$$V = \pi r^2 h;, \;\; \Delta V = V\sqrt{(\frac{\Delta h}{h})^2 + (2\frac{\Delta r}{r})^2}$$

4 Comparison of two values with uncertainties

In common life we always use our intuition when talking about somethings being "far" or "close" to each other. When it comes to comparing the results of measurements, only the uncertainty determines if two values are close (or identical) or not.

Comparison of the experimental result with a known value

Let's first consider the case when you want to compare the result of your experiment with some known and accepted value. For example, there is a postulated value for the speed of light $c = 299\ 792\ 458$ m/s, or very precisely measured value for the Planks constant $\hbar = 1.054\ 571\ 628(53) \times 10^{=34}$ J·c. In both examples the uncertainty for the reference value either does not exist, or is much smaller than the uncertainty of the experimental measurement.

Then in order to agree with each other the difference between a reference value x_{true} and am experimental value x_{exp} should be less than the experimental uncertainty Δx, or $x_{exp} - \Delta x \leq x_{true} \leq x_{exp} + \Delta x$. If the separation between two values is larger that the uncertainty - they disagree, probably due to some undetected systematic errors in the measurement procedure.

Notice that it does not matter how close two values look! For example, a measures speed of light of 2.995 ± 0.002 m/s does not agree with its reference value, even though it looks very close. At the same time, a measurement of 3.01 ± 0.02 m/s is within its uncertainty range from the true value, even though it is farther from it. This just means that the second experiment is much less accurate!

In order to further quantify the measurement precision, one should find the percent deviation from the reference value. This will tell how close the two values are. You can find the percent deviation from theory as follows:

$$\%\ deviation = \frac{|x_{exp} - x_{ref}|}{x_{ref}} \times 100\% \tag{9}$$

Comparison of two measured quantities with uncertainties

Suppose that the same physical parameter x has been measured using two different methods, and the results of both experiments (with their absolute uncertainties) are $x_1 \pm \Delta x_1$ and $x_2 \pm \Delta x_2$. We can say that these two independent measurements agree with each other if their uncertainty intervals intercept. For example, two measured values of the acceleration due to gravity $g_1 = 9.83 \pm 0.04$ m/s^2 and $g_1 = 9.75 \pm 0.07$ m/s^2 agrees within experimental uncertainties - since their uncertainty intervals overlap: $9.79 \leq g_1 \leq 9.87$ m/s^2 and $9.68 \leq g_2 \leq 9.82$ m/s^2.

In case of two experimental values one cannot say which one is more accurate, and the expression for the percent deviation is slightly different from Eq. 9:

$$\%\ deviation = \frac{|x_1 - x_2|}{\frac{x_1 + x_2}{2}} \times 100\% \tag{10}$$

5 Further reading

This write-up gave only the very basic introduction to the methods of error analysis and uncertainty calculations. If you see yourself doing research in science (not necessarily physics!) or engineering, you may find it wise to invest into a good book on data analysis, such as "An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements" by John R. Taylor.